Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

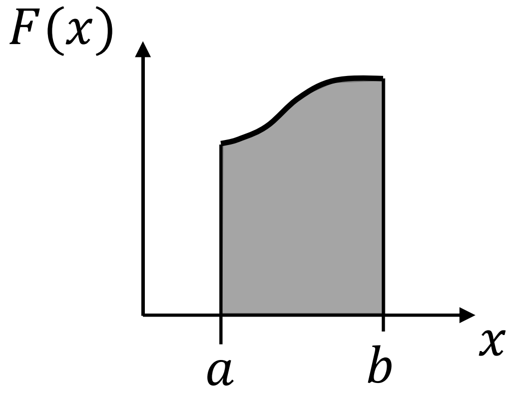
Finite element method (FEM)

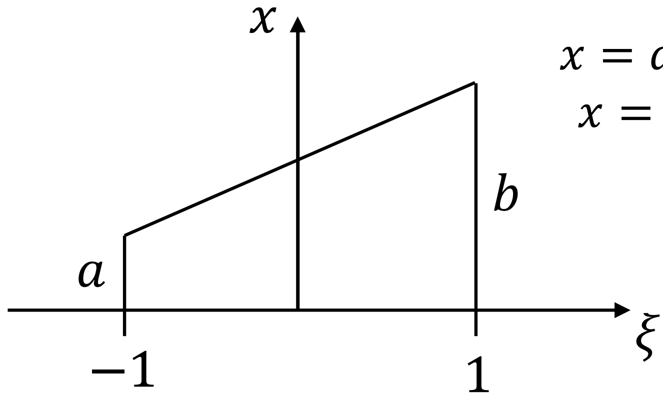
Numerical integration

03.2021

Defined integral

x − cartesian coordinate $\qquad\qquad$ ξ − natural coordinate





$$x = a \longrightarrow \xi = -1$$
$$x = b \longrightarrow \xi = 1$$

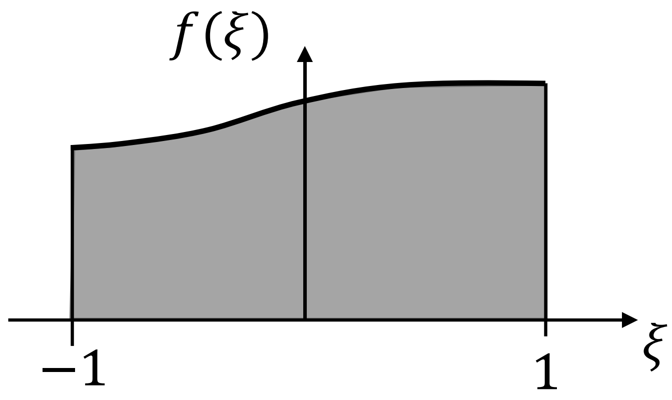
normalization of the function $F(x)$:

$$x(\xi) = \frac{b-a}{2}\xi + \frac{a+b}{2} \quad ; \quad f(\xi) = F\left(\frac{b-a}{2}\xi + \frac{a+b}{2}\right) \quad ; \quad dx = \frac{b-a}{2}d\xi$$

defined integral of the function $F(x)$:

$$\int_a^b F(x)dx = \int_{-1}^{1} f(\xi)\frac{b-a}{2}d\xi = \frac{b-a}{2}\int_{-1}^{1} f(\xi)d\xi$$

Gaussian quadrature rule



quadrature rule:

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{n} w_i \cdot f(\xi_i) + R_n$$

n – no. of sample points,
ξ_i – coordinates of sample points
w_i – weight coefficients
R_n – rest of the sum

$$R_n = 0 \quad \Rightarrow \quad \frac{d^{2n}f}{d\xi^{2n}} = 0$$

for linear function $f(\xi) = \alpha\,\xi + \beta$; $\frac{df}{d\xi} = \alpha$; $\frac{d^2 f}{d\xi^2} = 0$

$$2n = 2 \longrightarrow n = 1$$

$\int_{-1}^{1} (\alpha\,\xi+\beta)d\xi = w_1 \cdot f(\xi_1) + 0$; for one Gauss point: $\xi_1 = 0\,, w_1 = 2$

$$\int_{-1}^{1} (\alpha\,\xi+\beta)d\xi = w_1 \cdot f(0)$$

Gaussian quadrature rule

polynomial functions:

$2nd\ order$: $f(\xi) = \alpha\xi^2 + \beta\xi + \gamma\ ;\ \frac{df}{d\xi} = 2\alpha\xi + \beta\ ;\ \frac{d^2f}{d\xi^2} = 2\alpha\ ;\ \frac{d^3f}{d\xi^3} = 0$

$$2n = 3 \longrightarrow n = 1.5 \longrightarrow n = 2$$

for two Gauss points: $\xi_1 = -\frac{1}{\sqrt{3}}\ ;\quad \xi_2 = \frac{1}{\sqrt{3}}\quad ;\quad w_1 = w_2 = 1$

$$\int_{-1}^{1} (\alpha\xi^2 + \beta\xi + \gamma)d\xi = w_1 \cdot f\left(-\frac{1}{\sqrt{3}}\right) + w_2 \cdot f\left(\frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$3th\ order$: $f(\xi) = \alpha\xi^3 + \beta\xi^2 + \gamma\xi + \delta\quad ;\ \frac{d^4f}{d\xi^4} = 0 \longrightarrow n = 2$ (2 points)

$$\int_{-1}^{1} (\alpha\xi^3 + \beta\xi^2 + \gamma\xi + \delta)d\xi = w_1 \cdot f\left(-\frac{1}{\sqrt{3}}\right) + w_2 \cdot f\left(\frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Gaussian quadrature rule

$4th\ order$: $f(\xi) = \alpha\xi^4 + \beta\xi^3 + \gamma\xi^2 + \delta\xi + \varphi$; $\frac{d^5 f}{d\xi^5} = 0$

$$2n = 5 \longrightarrow n = 2.5 \longrightarrow n = 3$$

for three Gauss points: $\xi_1 = -\sqrt{0.6}$; $\xi_2 = 0$; $\xi_3 = \sqrt{0.6}$;

$$w_1 = w_3 = \frac{5}{9} \ ; w_2 = \frac{8}{9}$$

$$\int_{-1}^{1} (\alpha\xi^4 + \beta\xi^3 + \gamma\xi^2 + \delta\xi + \varphi)\, d\xi =$$

$$= \frac{5}{9} \cdot f\left(-\sqrt{0.6}\right) + \frac{8}{9} \cdot f(0) + \frac{5}{9} \cdot f\left(\sqrt{0.6}\right)$$

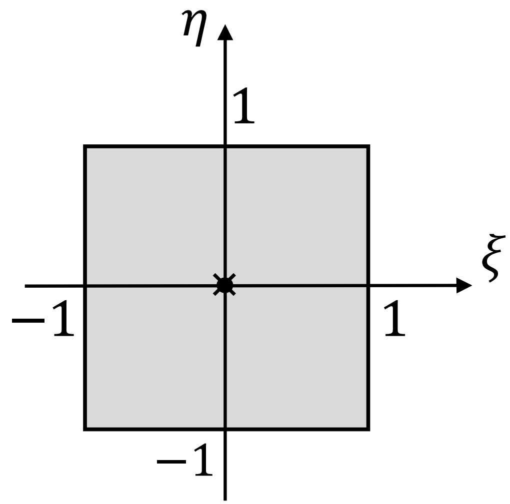
Gaussian quadrature rule for 2D FEs

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta = \int_{-1}^{1}\left(\sum_{i=1}^{n}(w_i \cdot f(\xi_i,\eta))\right) d\eta =$$

$$= \sum_{j=1}^{n} w_j \sum_{i=1}^{n}\left(w_i \cdot f(\xi_i,\eta_j)\right) = \sum_{j=1}^{n}\sum_{i=1}^{n}\left(w_i w_j \cdot f(\xi_i,\eta_j)\right)$$

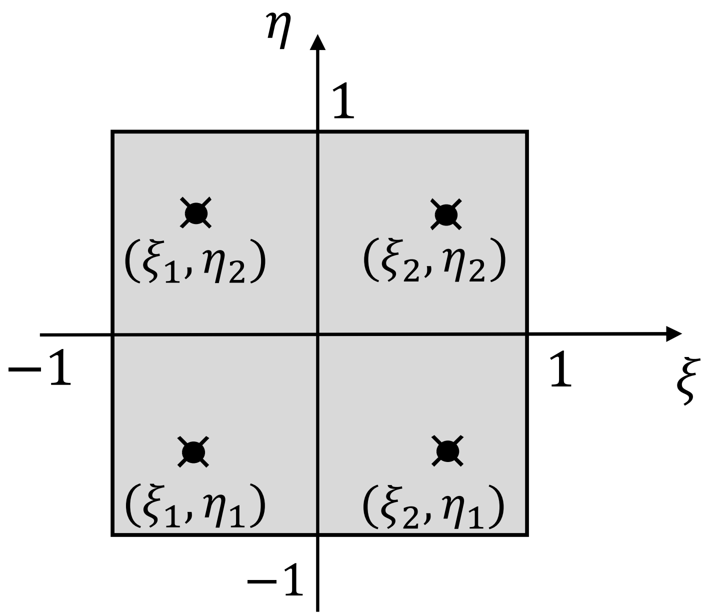
$n = 1$: $\xi_1 = \eta_1 = 0\,,\quad w_1 = 2$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta = w_1 w_1 \cdot f(0,0) = 4f(0,0)$$



Gaussian quadrature rule for 2D FEs

$n = 2$: $\xi_1 = \eta_1 = -\frac{1}{\sqrt{3}}$, $\xi_2 = \eta_2 = \frac{1}{\sqrt{3}}$; $w_1 = w_2 = 1$

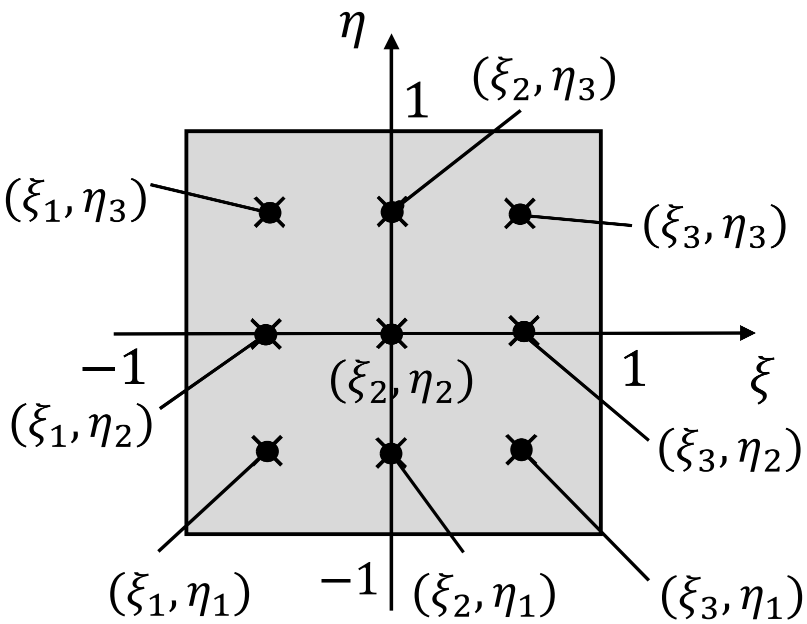


$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi d\eta = w_1 w_1 \cdot f(\xi_1,\eta_1) + w_2 w_1 \cdot f(\xi_2,\eta_1) +$$
$$+ w_2 w_2 \cdot f(\xi_2,\eta_2) + w_1 w_2 \cdot f(\xi_1,\eta_2) = f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) + f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) +$$
$$+ f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) + f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

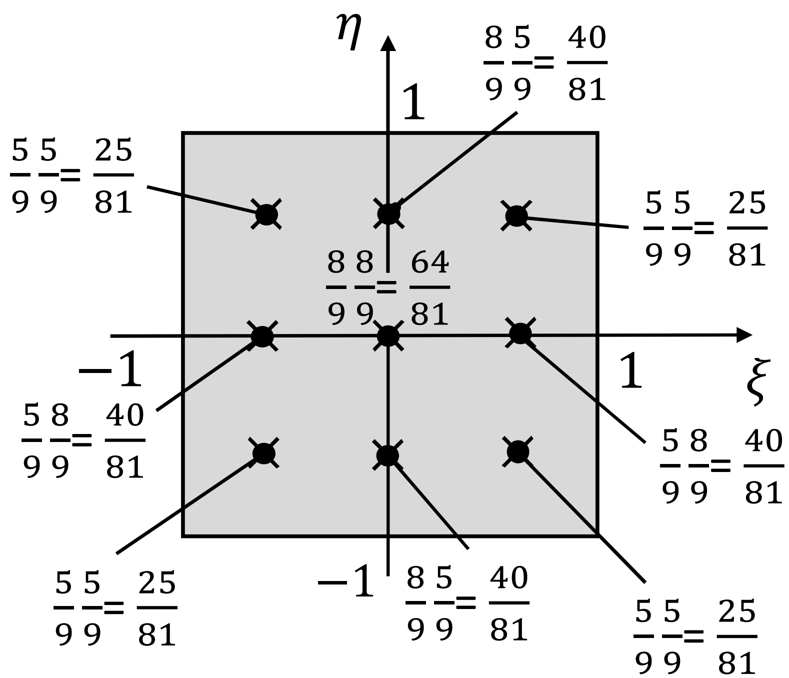
Gaussian quadrature rule for 2D FEs

$n = 3$: $\xi_1 = \eta_1 = -\sqrt{0.6}$, $\xi_2 = \eta_2 = 0$, $\xi_3 = \eta_3 = \sqrt{0.6}$

$$w_1 = w_3 = \frac{5}{9} \;\; ; \;\; w_2 = \frac{8}{9}$$



(ξ_i, η_j)



$w_i w_j$

Gaussian quadrature rule for 2D FEs

$n = 3$: $\xi_1 = \eta_1 = -\sqrt{0.6}$, $\xi_2 = \eta_2 = 0$, $\xi_3 = \eta_3 = \sqrt{0.6}$

$$w_1 = w_3 = \frac{5}{9} \quad ; \quad w_2 = \frac{8}{9}$$

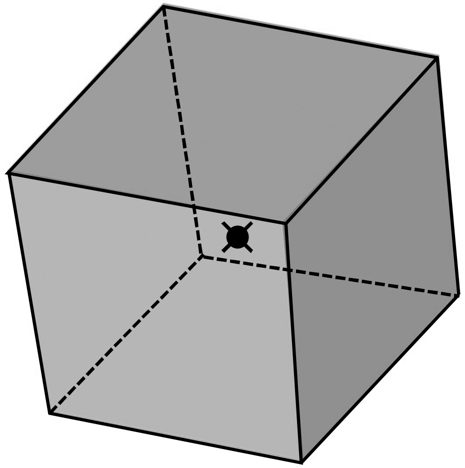
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta =$$

$$= w_1 w_1 \cdot f(\xi_1,\eta_1) + w_2 w_1 \cdot f(\xi_2,\eta_1) + w_3 w_1 \cdot f(\xi_3,\eta_1) +$$
$$+ w_1 w_2 \cdot f(\xi_1,\eta_2) + w_2 w_2 \cdot f(\xi_2,\eta_2) + w_3 w_2 \cdot f(\xi_3,\eta_2) +$$
$$+ w_1 w_3 \cdot f(\xi_1,\eta_3) + w_2 w_3 \cdot f(\xi_2,\eta_3) + w_3 w_3 \cdot f(\xi_3,\eta_3) =$$

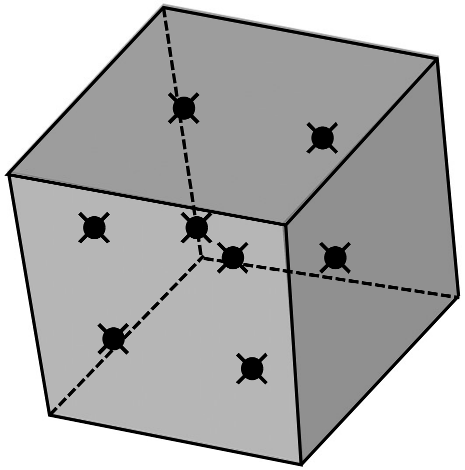
$$= \frac{5}{9}\frac{5}{9} f(-\sqrt{0.6}, -\sqrt{0.6}) + \frac{8}{9}\frac{5}{9} f(0, -\sqrt{0.6}) + \frac{5}{9}\frac{5}{9} f\left(\sqrt{0.6}, -\sqrt{0.6}\right) +$$
$$+ \frac{5}{9}\frac{8}{9} f(-\sqrt{0.6}, 0) + \frac{8}{9}\frac{8}{9} f(0,0) + \frac{5}{9}\frac{8}{9} f(\sqrt{0.6}, 0) +$$
$$+ \frac{5}{9}\frac{5}{9} f(-\sqrt{0.6}, \sqrt{0.6}) + \frac{8}{9}\frac{5}{9} f(0, \sqrt{0.6}) + \frac{5}{9}\frac{5}{9} f\left(\sqrt{0.6}, \sqrt{0.6}\right)$$

Gaussian quadrature rule for 3D FEs

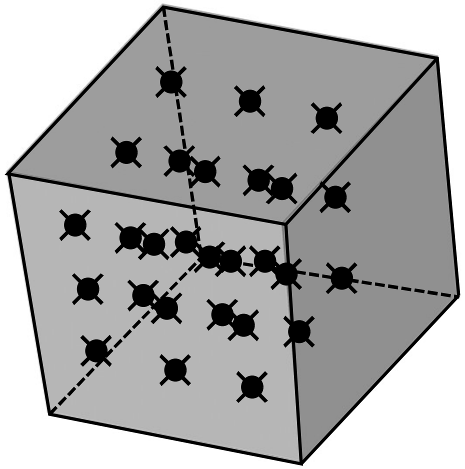
$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta\,)\,d\xi d\eta d\zeta = \sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(w_i w_j w_k \cdot f(\,\xi_i,\eta_i,\zeta_i)\right)$$



$n = 1$

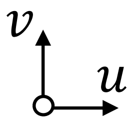
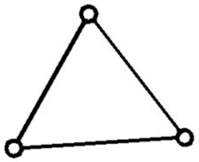
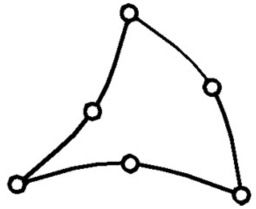
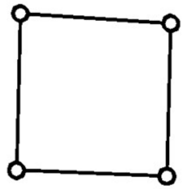
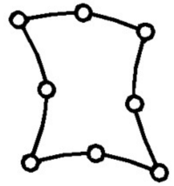


$n = 2\ \ (2 \times 2 \times 2)$

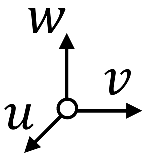
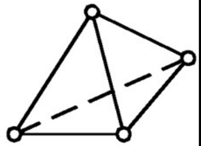
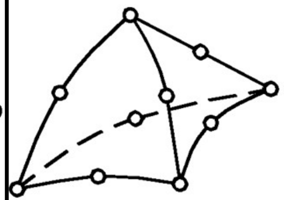
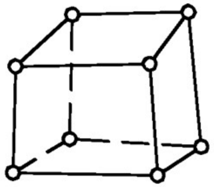
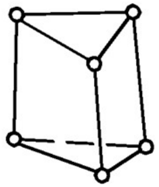
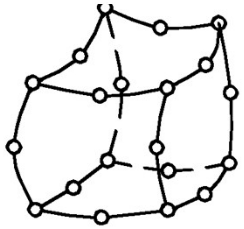


$n = 3\ \ (3 \times 3 \times 3)$

Integration schemes for 2D finite elements

 2D	3−node	6−node	4−node	8−node
Type of integration				
FULL	3	3	2×2	3×3
REDUCED	1	1	1	2×2

Integration schemes for 3D finite elements

 3D	4–node	10–node	8–node	6–node	20–node
Type of integration					
FULL	4	11	$2 \times 2 \times 2$	3×3	$3 \times 3 \times 3$
REDUCED	1	5	1	3×2	$2 \times 2 \times 2$